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Two-phase concave-type corner flows

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The flow of a two-phase system past an arbitrary corner has been studied by Healy (1970*a*), using the method of small perturbations, and some results were presented for convex-type flows. This paper, which is an extension of the above-mentioned one to concave-type flows, also compares the particle streamlines found from the perturbation with those obtained by numerically integrating the unperturbed equations. The agreement is found to be quite good for $\xi \approx 0{\cdot}2$ and excellent when $\xi = 0{\cdot}1$ or less, where $\xi = \alpha\tau$ is the particle parameter, α is proportional to the speed of the fluid and τ is the particle relaxation time. The nature of concave corner flows abruptly changes when the angle β through which the flow is deflected is 90°. For $\beta < 90°$, all particles collide directly except those approaching on streamlines near the stagnation line. When $\beta = 90°$ the critical value of ξ is $\xi_c = 0{\cdot}25$ and, for $180° > \beta > 90°$, only particles approach ing on streamlines near the stagnation line all collide. No particle-free zones exist in concave-type flows and the particle density increases monotonically in the downstream direction along all particle streamlines. The approximate effects of viscosity are also discussed.

1. Introduction

The flow of an inviscid incompressible fluid with embedded small spherical particles past an arbitrary corner has been studied by the author (1970*a*), using the method of small perturbations. The results, which are valid for small initial particle density k_0, were confined to convex-type flows and an approximate fluid-separation model was also considered.

The purpose of this paper is: to extend the above-mentioned work to concave-type flows; to compare the particle streamlines found by the perturbation method with those obtained by numerically integrating the unperturbed particle momentum equations; to study the critical collision conditions and the approximate viscous effects.

The perturbed equations, and other references thereto, are given in the first paper and the solutions are here adopted with little further discussion. A new assumption of non-sedimentation must now be added; i.e. if a particle collides with a wall it is assumed absorbed by it and thus removed from the flow.

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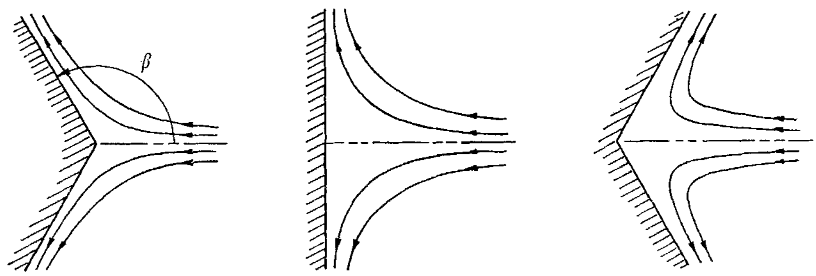
2. The results of the perturbation

The unperturbed fluid velocity is

$$\mathbf{u}_0 = -\alpha r^{n-1} \cos n\phi \, \mathbf{r}_0 + \alpha r^{n-1} \sin n\phi \, \boldsymbol{\phi}_0, \tag{1}$$

where α is a large constant, r and ϕ are the co-ordinates, $\mathbf{r}_0$ and $\boldsymbol{\phi}_0$ are unit vectors, and n is the geometrical parameter π/β, with β the angle between the wall and the stagnation line. The values of n and β for the geometries here considered are given in figure 1.



(a) (b) (c)

FIGURE 1. Flow configurations. (a) $n = \frac{3}{2}$, $\beta = \frac{2}{3}\pi$; (b) $n = 2$, $\beta = \frac{1}{2}\pi$; (c) $n = 3$, $\beta = \frac{1}{3}\pi$.

The equation governing the particle density was found to be

$$-[\cos n\phi + \xi(n-1) r^{n-2}] \frac{\partial k}{\partial r} + \frac{\sin n\phi}{r} \frac{\partial k}{\partial \phi} = 2\xi(n-1)^2 r^{n-3} k, \tag{2}$$

where k is the ratio of the particle to fluid mass densities and $\xi = \alpha\tau$, with τ the particle relaxation time. On using the method of characteristics, (2) yields one set of characteristic surfaces that is coincident with the particle streamlines and an integral surface that is represented by the variation of the particle density along the set. The latter, and the particle streamlines, are defined by

$$r^{2-n} \sin^{n^{-1}(2-n)} n\phi - (n-2)(n-1)\, \xi \int \sin^{2(n^{-1}-1)} n\phi \, d\phi = c \quad (n \neq 2) \tag{3}$$

and

$$r^2 \sin 2\phi \tan^{\xi} \phi = c \quad (n = 2), \tag{4}$$

where c is the streamline constant. The integral surface of (2) is given by

$$\frac{k}{k_0} = \left[1 + (n-2)(n-1) \frac{\xi}{c} \int \sin^{2(n^{-1}-1)} n\phi \, d\phi\right]^{2(n-1)/(n-2)} \quad (n \neq 2) \tag{5}$$

and

$$k/k_0 = \tan^{\xi} \phi \quad (n = 2). \tag{6}$$

From (3), the co-ordinate of a particle streamline is

$$r = [c + (n-2)(n-1)\, \xi \int \sin^{2(n^{-1}-1)} n\phi \, d\phi]^{1/(2-n)} / \sin^{n^{-1}} n\phi \tag{7}$$

and the innermost particle streamlines, or zone delineation curves, are obtained when $c = 0$. So long as $n < 1$ these curves are spiral-shaped; they start at the upstream side of the corner, partially encircle it and trail downstream, thus delineating a single-phase zone. In the range $1 < n < 2$, the sign on the integral in (7) is negative and hence no real particle-free zone exists. Moreover, when

$n > 2$, the μ-theorem shows that the integral diverges if either of the limits $\phi = 0$ or π/n is included in the range of the integration and, in addition, the exponent $1/(2-n)$ is negative. The implication is that no finite particle-free zone exists and that it is necessary to assume that the initial position of the particle is (r_0, ϕ_0), where $\phi_0 > 0$.

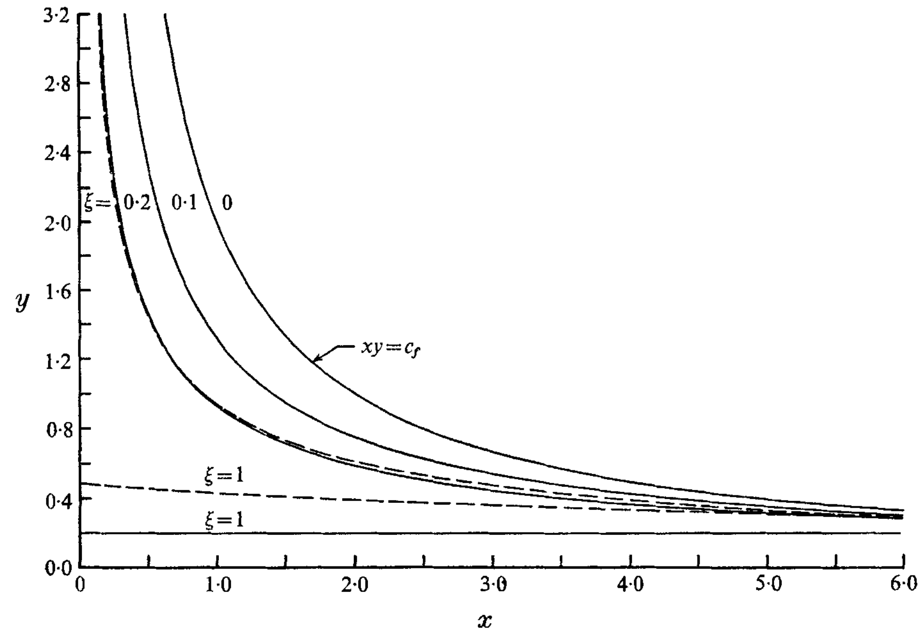


FIGURE 2. Particle streamlines ($n = 2$, $c_f = 2$; $x_0 = 10$). ——, perturbation theory; – – –, exact.

In all concave flows, the fluid and particle velocities increase rapidly with distance from the corner and, since the interest of this paper is in the flow near the latter, such large velocities are ignored and it is assumed that the fluid and particles have a common velocity at the initial co-ordinate (r_0, ϕ_0).

Evaluation of (3) for $n > 2$ shows that almost all particles collide directly and seem very little influenced by the turning motion of the fluid. Those most strongly influenced approach on streamlines near the stagnation line and further study is deferred until § 4.

If the particle streamlines in stagnation point flow are plotted directly from (4), the particles with larger values of ξ approach the corner on streamlines nearer to the line of symmetry. For purposes of comparison, it is desirable that all particles approach on the same streamline, which necessitates choosing the streamline constant c as a function of ξ and the initial co-ordinate x_0. In the Cartesian system, the result is

$$y = c_f x_0^{-2\xi/(\xi+1)} x^{(\xi-1)/(\xi+1)}, \tag{8}$$

where c_f is the constant of the fluid streamline, defined by $xy = c_f$, along which the particles approach. This relation has been used to compute the streamlines shown in figure 2. As expected, when $\xi = 1$ the perturbation theory is a poor approximation and the result is a straight line y = constant. The next largest value of ξ here used is 0·2 and, from the shape of the trajectory, this is close to the critical value, which is discussed in § 4. The particle density distribution for $n = 2$ was computed using (6), which is independent of radius, and plotted in figure 3. The density gradually increases with angle, being zero on the stagnation line and increasing to infinity on the wall. It is also clear that the particle density

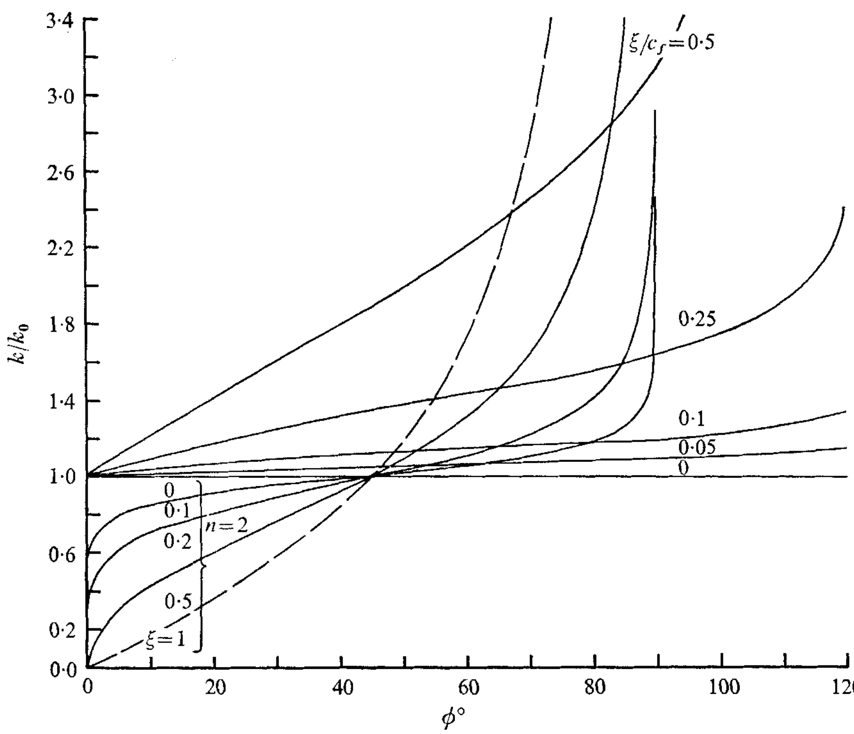


FIGURE 3. The particle density distributions ($n = 2, \frac{3}{2}$. $c_f = 2$).

is not uniform as the fluid approaches the corner. In an earlier paper, Healy (1970*b*) has studied the flow of the same type of system past a flat plate of finite width standing normal to the approaching stream. In this case, the particle density remains almost uniform until a distance of about one plate width from the stagnation point is reached. Thereafter, the density increases rapidly but does not become infinite.

For the flow past a wedge, (3) and (5) are easily evaluated by using Simpson's rule. However, from the previous study, the author has had a Runge–Kutta computer program available for a general corner and has used this instead to evaluate the ordinary differential equations derived from (2). The particle streamlines for $n = \frac{3}{2}$ are shown in figure 4 and it is evident that the magnitude of the perturbation has decreased relative to that for $n = 2$. It is expected that the perturbation gradually becomes smaller as $\beta \to 180°$. The particle density increases monotonically along the particle streamlines in the downstream direction as shown in figure 3; the amount of increase decreases rapidly if streamlines at a greater distance from the origin are chosen, i.e. greater values of c_f.

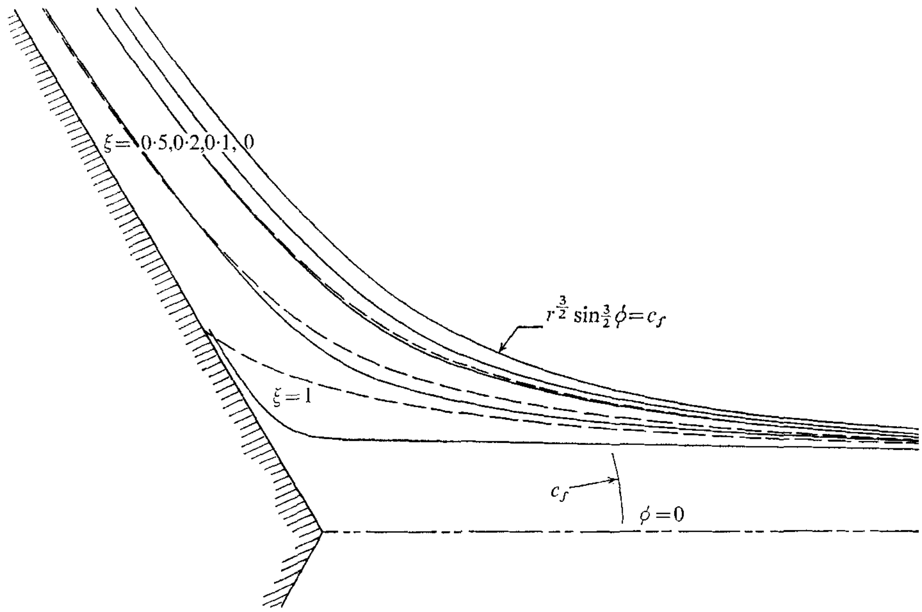


FIGURE 4. Particle streamlines ($n = \frac{3}{2}$, $c_f = 2$, $r_0 = 10$). ——, perturbation theory; – – –, exact.

3. Some exact results

The unperturbed particle momentum equations for small k_0 are

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{u_x - v_x}{\tau},$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \frac{u_y - v_y}{\tau}.$$

An examination of this system shows that the particle streamlines $dy/dx = v_y/v_x$ form characteristic surfaces that are common to both equations. Using this relation to eliminate y from the independent variables yields

$$\tau v_x (dv_x/dx) = u_x - v_x, \quad \tau v_x (dv_y/dx) = u_y - v_y. \tag{9}$$

Even in the simplest case of stagnation point flow, when $u_x = -\alpha x$ and $u_y = \alpha y$, the set of equations (9) is not analytically solvable; the first equation may be solved exactly, but v_x is found in an implicit form only. On the other hand, the system is not difficult to solve numerically; it has already been done by Michael & Norey (1969). For general corner flow, the system is best constructed in the following parametric form:

$$\left.\begin{aligned} \xi(dv_x/dt) &= -v_x - (x^2+y^2)^{\frac{1}{2}(n-1)} \cos\left[(n-1)\tan^{-1}(y/x)\right], \\ \xi(dv_y/dt) &= -v_y + (x^2+y^2)^{\frac{1}{2}(n-1)} \sin\left[(n-1)\tan^{-1}(y/x)\right], \\ (dx/dt) &= v_x, \quad (dy/dt) = v_y, \end{aligned}\right\} \tag{10}$$

where $\xi = \alpha\tau$ and each velocity is now divided by α.

It is assumed that the particle and fluid have the same velocity at some initial point (x_0, y_0) and, with this as an initial condition, the system (10) has been evaluated for $n = \frac{3}{2}$ and 2 using Runge–Kutta procedures, and the results are shown in figures 2 and 4. In general, for $\xi = 0{\cdot}1$ or less, the results are coincident with those of the perturbation theory and, for $\xi = 0{\cdot}2$, for most practical purposes, the difference is negligible. Equation (4) provides an excellent example of perturbation theory at its best; this very simple relation yields results (see figure 2) that, for ξ near $0{\cdot}1$ or less, are as good as those of equations (10).

4. The critical collision conditions

These conditions may partially be established by considering the manner in which the particle approaches the corner along the stagnation line. The particle momentum equation along this line is

$$\xi v(dv/dx) + v + x^{n-1} = 0, \tag{11}$$

and is of the Emden type, where $\xi = \alpha\tau$ and v is now replaced by v/α.

For stagnation point flow $n = 2$ and the equation may be solved exactly. An analysis by Healy (1970*b*) of its behaviour near the singular point showed the critical value of ξ to be $0{\cdot}25$. A detailed account of this type of analysis is given by Michael & Norey. It is also found that ξ_c is independent of the initial co-ordinate x_0 of the particle.

The other ranges of n that are of interest are $2 < n < \infty$ and $1 < n < 2$. For some values of n in the former range (11) is also of interest in astrophysics and, for this reason, has received some attention. Fowler (1931) has established the existence and uniqueness of its solutions and also has studied the asymptotic solutions in the limit $x \to 0$. Although the ranges of the variables and the initial conditions in the astrophysics case are considerably different from those in the present one, the analysis may readily be adapted.

Equation (11) may be expressed in the following parametric form:

$$d\eta/dt + \eta + \nu^{n-1} = 0, \quad d\nu/dt = \eta, \tag{12}$$

where
$$\nu = x\xi^{1/(n-2)} \quad \text{and} \quad \eta = v\xi^{(n-1)/(n-2)}/\alpha \tag{13}$$

and the initial conditions are

$$t = 0, \quad \nu = \nu_0, \quad \eta = -\nu_0^{n-1} = \eta_0.$$

The locus of all such initial points (ν_0, η_0) in the phase plane provides envelopes for the solution curves, as shown in figure 5; the region under the envelope being the one of interest. In the limit $\xi \to 0$, these envelopes are also the trajectories for particles starting at any point (ν_0, η_0) with the prescribed initial conditions. For a given n (> 2), as the initial co-ordinate ν_0 is increased, the trajectory for finite ξ deviates increasingly from this envelope; two such trajectories are shown for $n = 4$. Ultimately, a point (ν_c, η_c) is reached from which the trajectory becomes asymptotic to the line $\eta = -\nu$, in the limit $\nu \to 0$. This trajectory, which was called the 'E-solution' by Fowler, is, in the present case, the 'critical trajectory'.

Particles for which $\nu_0 > \nu_c$ reach the origin with a finite velocity, i.e. collide, and those for which $\nu_0 \leqslant \nu_c$ require an infinite time to collide. Using the asymptotic property of the solutions established by Fowler, the critical trajectories, or E-solutions, were found by numerically integrating (12) backward in time; the results for $n = \frac{5}{2}$, 3, $\frac{7}{2}$ and 4 are shown in figure 5. The critical points are $\nu_c = x_c \xi_c^{1/(n-2)}$ and, therefore, one may consider that a critical co-ordinate x_c exists, for a given ξ, or that a critical $\xi = \xi_c$ exists for a given x_0.

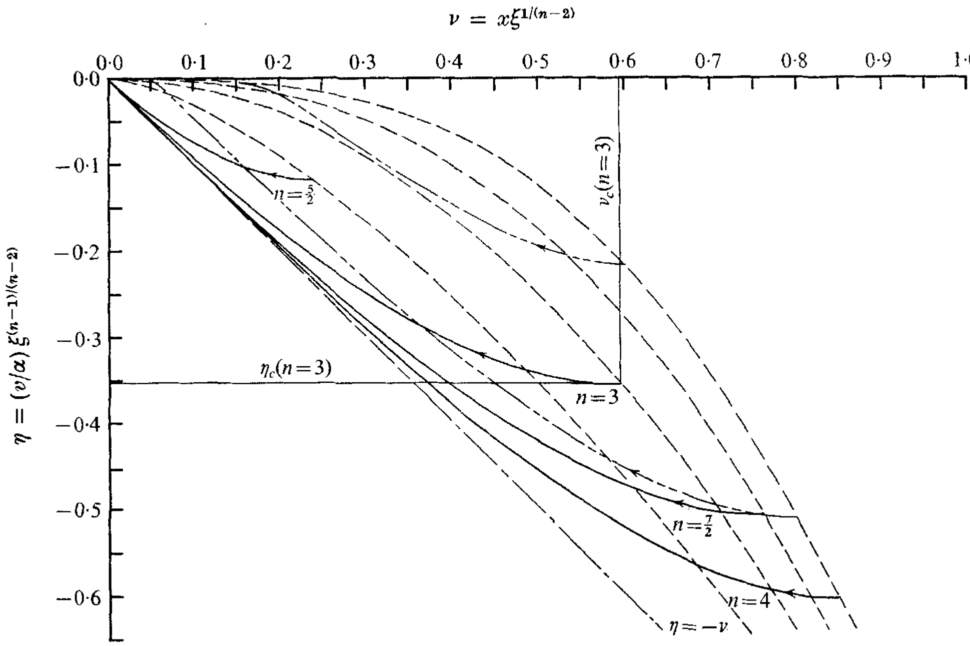


FIGURE 5. The critical and other trajectories in phase plane. ——, critical trajectories; – – –, envelopes $\eta = -\nu^{n-1}$; — - - —, non-critical trajectories.

If (10) are transformed, using (13), numerical integrations show that the above results are valid for particles approaching along streamlines near the stagnation line. Particles approaching on streamlines somewhat further away seem unaware of the turning motion of the fluid and collide rapidly. Some trajectories for $n = 3$ are shown in figure 6. When $\nu < \nu_c$ and $\zeta \leqslant 0{\cdot}1$ approximately, these trajectories all seem either to intersect with the downstream wall or to become asymptotic to it.

Before considering the particle trajectories in the range $1 < n < 2$, it is instructive to study the behaviour of (11) near the singular point in the limit $\xi \to 0$. In this case $v = u = dx/dt = -x^{n-1}$ and consequently

$$t - t_0 = -\int_{x_0}^{x} \lambda^{1-n}\, d\lambda.$$

When $n = 2$, an infinite time is required to reach the origin, but for $n < 2$, in the limit $x \to 0$, t goes to a finite value always. Furthermore, since $|v| \geqslant |u|$ for ξ

finite, all particles approaching on streamlines near the stagnation line collide near the apex of the wedge. On the other hand, it has been seen in figure 4 that particles approaching along streamlines further away do not all collide. Integrations of the transformed system (10) for $n = \frac{3}{2}$ show that when $\zeta_0 = y_0/\xi^2 \leqslant 1{\cdot}0$ roughly, all

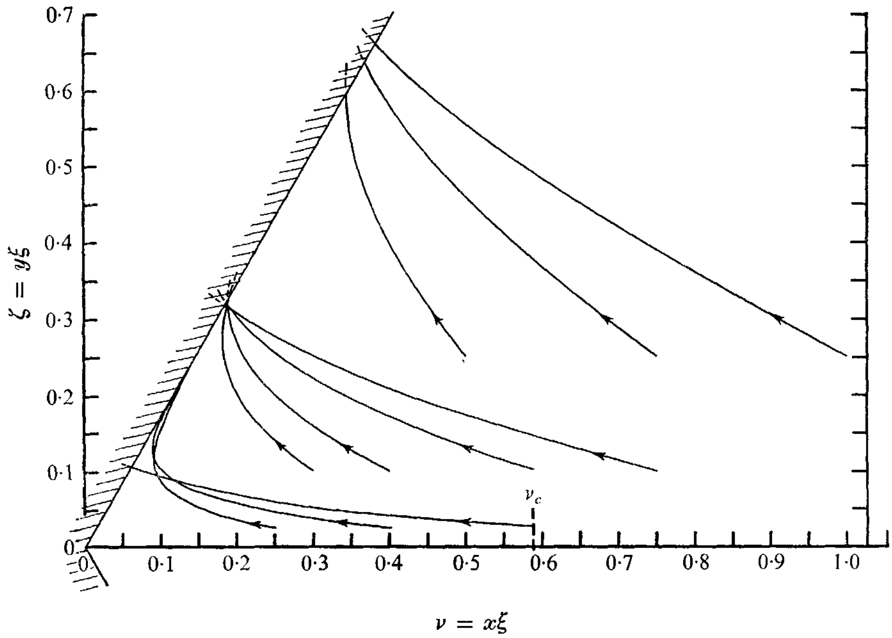


FIGURE 6. Some particle trajectories for $n = 3$.

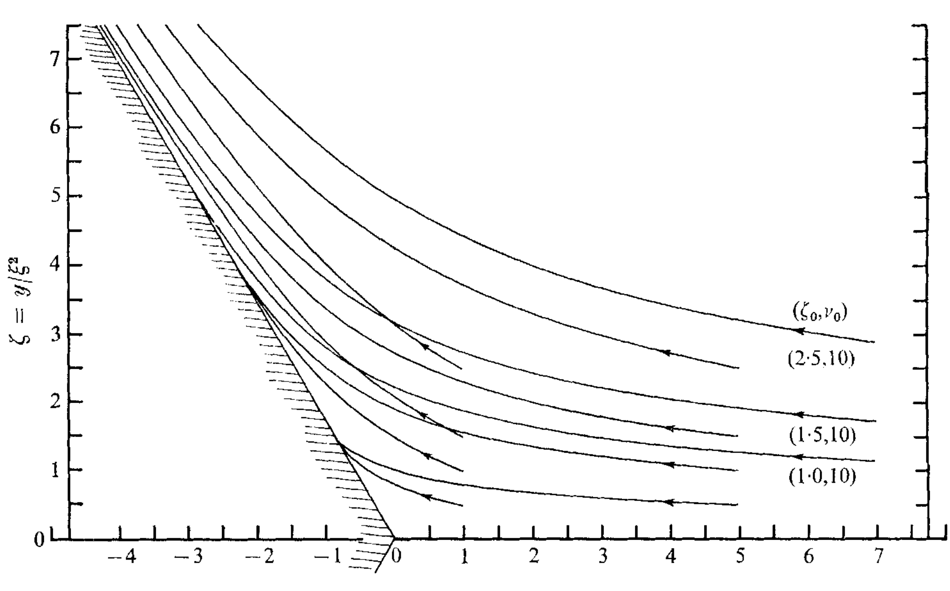


FIGURE 7. Some particle trajectories for $n = \frac{3}{2}$.

the particles collide. The greater the value of ζ_0, the further downstream the particle collides. Trajectories with various (ζ_0, ν_0) are shown in figure 7; for ζ_0 greater than about 2, the particles do not seem to collide.

5. The approximate viscous effects

In most of this paper a high-speed flow has been assumed, and a boundary layer must be considered if the fluid is viscous and $R_x \gg 1$, where R_x is the Reynolds number based on the distance from the corner. A further requirement is that the length scale on which the fluid changes occur must be much greater than the particle dimensions, i.e. $d/x \ll (R_x)^{\frac{1}{2}}$, where d is the particle diameter. These requirements are satisfied by a wide range of flows.

The particle may enter the boundary layer at the stagnation point or elsewhere and, in general, moves under some combination of the following forces: (i) the Stokes drag, which has no component directed away from the downstream wall, (ii) the inertia of the particle when it enters the boundary layer away from the stagnation point and (iii) the force exerted across the streamlines by the relative velocity between the particle and the fluid. Neglecting the third force means that there is no way by which the particles can leave the boundary layer, once they enter it, and the net effect is sedimentation. The third force, although small, can have a sedimenting or desedimenting effect on the particles and, although it has been much investigated since it was reported by Segré & Silberberg (1962), its precise nature and magnitude have not yet been determined but, on the basis of experimental evidence, one may qualitatively infer that its effect will be that of desedimenting the particles near the stagnation point.

6. Summary

The perturbed equations for small initial particle density have been analytically solved in a previous paper and the results for concave-type corner flows are here evaluated using simple numerical procedures.

Concave flows have no particle-free zones and their nature, in general, abruptly changes at $\beta = 90°$. For $\beta < 90°$ numerical integrations show that, with the exception of those that approach on streamlines near the stagnation line, all particles seem unaffected by the turning motion of the fluid and collide directly.

When $\beta = 90°$, the critical value of ξ is found to be $\xi_c = 0{\cdot}25$ so that, for $\xi < 0{\cdot}25$, the particles should not collide whereas, for $\xi > 0{\cdot}25$, they should.

In the range $180° > \beta > 90°$, there are no sudden changes and the perturbation effects become gradually smaller as $\beta \to 180°$. All particles approaching the origin on streamlines near the stagnation line collide in the vicinity of the apex of the wedge and the critical collision conditions then involve ξ and the initial particle co-ordinates. For $n = \frac{3}{2}$, particles with initial y co-ordinate within ξ^2 approximately of the stagnation line collide near the apex and those with initial co-ordinate greater than $2\xi^2$ roughly do not seem to collide. Those approaching on streamlines between these limits collide at varying distances along the downstream wall, or become asymptotic to it.

As is characteristic of convex-type flows, the particle density increases monotonically in the downstream direction along the particle streamlines in concave flows also.

A particle moving in a shear flow with a velocity relative to that of the fluid, experiences a force normal to the streamlines, which is not yet fully understood. This force is expected to cause some desedimentation in the boundary layer but, generally, the viscous effects are expected to promote sedimentation.

The accuracy of the perturbation theory has been checked by a numerical integration of the unperturbed equations. For $\xi \approx 0{\cdot}1$, the agreement is excellent, whereas, for $\xi \approx 0{\cdot}2$, it is quite good.

The Runge–Kutta procedures used by the author have been carried out on an IBM 7040 digital computer using a fourth-order program with an automatically adjustable step. Whenever possible, the program was checked using the limiting case of $\xi \to 0$, whose exact solution generally is known.

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